Business cycles and on the job search∗

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Abstract:

We analyse the steady state and business cycle properties of a stochastic version of Burdett and Mortensen (1998) on-the-job search model with heterogeneous firms. The model is solved using a novel numerical method, projection within perturbation, which uses Chebyshev polynomial approximation and Clenshaw-Curtis quadrature for dealing with heterogeneity. In the model we assume wage bargaining mechanism for wage setting instead of wage posting of Moscarini and Postel-Vinay (2016). We analyse how this change influences model properties such as worker flows between firms, distribution of firm size and wages, and the dynamics of unemployment, wages and vacancies under productivity and other shocks.

Keywords: job search, business cycle, heterogeneous firms, unemployment, computation method

Journal of Economic Literature Classification Numbers: J64, E32, J31

1 Introduction

In their seminal article Burdett and Mortensen (1998) (henceforth, BM) provide a simple answer to the question of why workers of similar skills are paid different wages for the same kind of work. They show that in a simple model where firms commit to wage offers and in which workers search on-the-job and move from lower to higher paying jobs there

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exists a unique and stable equilibrium wage distribution. Surprisingly, this result holds for both the case of identical and heterogeneous workers and firms. The BM model has since become a workhorse model for theoretical and empirical labour market economists analysing wage inequality, labour turnover and the role of on-the-job search.

In this paper we construct a stochastic Burdett and Mortensen (1998) on-the-job-search model based on recent work by Moscarini and Poster-Vinay (2013, 2016), henceforth referred to as MPV. We consider the case with exogenously heterogeneous firms which differ in their individual productivity. Solving the model with heterogeneous firms and aggregate shocks is non-trivial, due to the size of the state vector. In the model the set of state variables includes a distribution function, which is an infinite-dimensional object. To solve the model we propose a two-step approach which allows to circumvent most of computational problems. First, we approximate all the functions of individual firm productivity using Chebyshev polynomials. It reduces the problem to a finite set of state variables (weights) and allows us to use Clenshaw-Curtis quadrature to calculate integrals which appear in the model. In the next step, we approximate the dynamics in the model using perturbation method (in our case we linearise the model around the steady state) and track the evolution of Chebyshev nodes under these dynamics. Using this approach we can efficiently solve for the stochastic equilibrium and study the model properties, e.g. the distribution of firms, wages, reallocation of workers across employers and dynamics of labour market variables. We also analyse effects of aggregate shocks and the performance of the model under alternative parameters.

We follow closely the approach and parametrization utilized by MPV, but we change one key assumption regarding labour market transitions and consequently wage setting. Instead of having employed workers switch jobs when they find an alternative one in a firm offering a higher wage, we assume that they switch jobs to firms of higher productivity. Regarding wage setting, we assume that a worker and firm negotiate how to split the surplus of a match according to the Nash wage bargaining mechanism. This assumption has certain advantages. First, it allows us to avoid the critique put forth by Shimer (2006) regarding the inapplicability of Nash wage bargaining in a model incorporating on-the-job search. In our model when setting wages firms do not take into account their possible allocative effect. It is not possible that a higher wage than the one negotiated could increase firms’ profits due to higher retention of own workers and poaching of workers from other firms. Furthermore, using wage bargaining allows us to solve the model using the proposed method. Finally, in this set-up, firms of higher productivity will always offer higher wages, so that ex-post workers will always go up the pay ladder.

We find that our model does a good job in replicating the empirical patterns of gross and net job creation, a key point highlighted in Haltiwanger et al. (2014) and Moscarini
and Postel-Vinay (2012). Moreover, we manage to address issues that wage posting models fail to replicate: when wages no longer play an allocative role, our model produces much greater equilibrium wage dispersion comparing to the wage posting model and we can easily replicate what is found in the data. On the other hand, using Nash wage bargaining lowers the model’s amplification of shocks, a feature which has already been documented in the literature in Andolfatto (1996) or Costain and Reiter (2008) and results in lower volatility of the job finding rate.

The rest of the paper is organized as follows. Section 2 presents the on-the-job search model. In Section 3 we describe the two-step solution method we employ. Section 4 contains the parametrization and the results, while Section 5 concludes.

2 Model

We consider a dynamic version of the labour market model of Burdett and Mortensen (1998), following Moscarini and Postel-Vinay (2016) specification. The basic setup of the model is as follows. Time is discrete and we calibrate the model so that one period is equivalent to a quarter. The economy is composed of a unit measure of workers and we assume that both unemployed and employed household members randomly search for jobs. There is a unit measure of firms which differ in their productivity according to a continuous strictly positive distribution function. To hire workers firms post vacancies. In equilibrium, upon sampling a vacancy in a firm, unemployed commence employment, whereas employed will only switch jobs in case of matching with a firm of higher productivity. Firm uses labour and its output depends on individual firm productivity and a stochastic aggregate economy-wide component. Finally, workers and firms set the wage according to Nash wage bargaining.

2.1 Household

Consider an economy that is populated by a unit mass of workers and that in each period an individual worker can be either employed or unemployed. Both employed and unemployed workers search for a job and from firms’ perspective they are perfect substitutes. We assume that an unemployed receives utility $b$ and search for jobs with a constant intensity $\lambda_u$, whereas an employed worker searches on the job with constant intensity equal to $\lambda_e$. All jobs are destroyed with an exogenous probability of $\delta$.

2.2 Firm

There is a unit mass of firms that are heterogeneous with respect to their individual productivity level. A type $p$-firm hires labour and produces output using a linear production
function with productivity equal to $A_t p$, where $A_t$ is aggregate productivity level in period $t$ which follows a standard AR(1) process.\footnote{In the following exposition, we will use the phrase type $p$-firm and a firm of productivity $p$ interchangeably, despite the fact that when $A_t$ is not equal to one the two do not mean exactly the same.} Productivity is distributed according to a cumulative distribution function $\Gamma$ with support on the interval $[p, \bar{p}]$. We assume that the probability density function given by $\gamma = \Gamma'$ is strictly positive and continuous on the support. In order to hire workers firms must post vacancies $V_t(p)$, for which they incur an increasing convex cost $c(V)$.

We can express the total number of posted vacancies $V_t$ as

$$V_t = \int_{p}^{\bar{p}} V_t(p) \gamma(p) dp,$$

where $V_t(p)$ denotes the number of vacancies posted by a firm of productivity $p$ at time $t$.

The share of total vacancies posted by firms of productivity higher than $p$, $\mathcal{V}_t(p)$, and the share of vacancies posted by firms of productivity no greater than $p$, $\mathcal{V}_t(p)$, can be computed as

$$\mathcal{V}_t(p) = \int_{p}^{\bar{p}} V_t(p) \gamma(p) dp \quad \mathcal{V}_t(p) = \int_{p}^{\bar{p}} V_t(p) \gamma(p) dp$$

\section*{2.3 Labour Market}

Let $N_t(p) : [p, \bar{p}] \to [0, 1]$ be the cumulative distribution function of employed workers at firms of different productivity in period $t$. Define total employment, $N_t$, as $N_t = N_t(\bar{p})$ and unemployment rate, $U_t$, as $U_t = 1 - N_t$. The average size of a firm of productivity $p$, $L_t(p)$, can be written as:

$$L_t(p) = \frac{dN_t(p)/dp}{\gamma(p)}.$$  

The mass of unemployed and employed workers that search for job, $O_t$, is matched with vacancies posted by firms, $V_t$, according to aggregate matching function:

$$M_t = \nu V_t^{\mu} O_t^{1-\mu}.$$  

The number of job applications sent by unemployed $O_t^{U}$ and employed $O_t^{N}$ members of the household are given by:

$$O_t^{U} = U_t \lambda_u \quad O_t^{N} = N_t \lambda_e,$$
giving the total number of job offers as:
\[ O_t = O_t^U + O_t^N. \] (6)

Using above values we can determine how many of the job matches are of persons who were previously unemployed \( M_t^U \) and employed \( M_t^N \):
\[ M_t^U = \frac{O_t^U}{O_t} M_t \quad M_t^N = \frac{O_t^N}{O_t} M_t \] (7)

Aggregate employment evolves according to:
\[ N_t = (1 - \delta) N_{t-1} + M_t^U, \] (8)

where \( \delta \) is the exogenous job separation rate.

We assume that all job matches of previously unemployed agents are productive and that such job seekers always take up employment.\(^2\) The same is not true for on-the-job matches - only those workers who find a higher productivity firm will switch jobs. The Nash wage bargaining procedure that is used for setting wages ensures that firms of higher productivity post higher wages. We can express the probability for an unemployed person to find a job, \( \Phi_t^U \), and the probability that an on-the-job seeker meet a new employer, \( \Phi_t^N \), as
\[ \Phi_t^U = \frac{M_t^U}{U_t} \quad \Phi_t^N = \frac{M_t^N}{N_t} \] (9)

Similarly, the probability for an employer to meet any job-seeker (per vacancy) is given by
\[ \Psi_t = \frac{M_t}{V_t}, \] (10)

and probabilities for a firm to meet a previously unemployed and employed job seeker equal
\[ \Psi_t^U = \frac{M_t^U}{V_t} \quad \Psi_t^N = \frac{M_t^N}{V_t}, \] (11)

respectively.

Using the above notation we can now discuss the dynamics of the distribution of workers across firms. Note that employed job seekers will switch jobs only if they have are offered a higher than currently receiving wage. In equilibrium, given the assumption of

\(^2\)In order for this to be the case, the parameters of the model must result in respective value functions (which are described later on) being positive for all firm productivity.
Nash wage bargaining with constant bargaining power for wage setting, which is discussed in greater detail later on, firms of higher productivity offer higher wages. The average size of a type-\(p\) firm evolves as follows:

\[
L_{t+1}(p) = (1 - \delta) \left( 1 - \Phi_t^N \frac{V_t(p)}{V_t} \right) L_t(p) + U_t \Phi_t^U \frac{V_t(p)}{V_t} + (1 - \delta) N_t \Psi_t^N \frac{V_t(p) N_t(p)}{V_t N_t}. \tag{12}
\]

The first term of equation (12) describes workers that do not change their job due to either exogenous separation or better job offer. Recall that an exogenous fraction \(\delta\) of workers is exogenously separated from their job and a fraction \((1 - \delta)\) does not become unemployed. The probability for a worker to meet another employer equals \(\Phi_t^N\). Since the worker quits his current job only if the job offer comes from a firm of higher productivity. Hence the probability that a worker receive a better-paying offer is \(\Phi_t^N V_t(p) V_t\).

The second term describes flows of currently unemployed workers. The probability of employing a previously unemployed job-seeker is given by \(U_t \Phi_t^U\), and this probability must be corrected to account for the intensity of vacancy posting by the type-\(p\) firm. The last term relates to the "poaching" behaviour of a firm. The probability for a type-\(p\) firm of meeting a random previously employed job-seeker is \(N_t \Phi_t^U \frac{V_t(p) V_t}{V_t^N}\). This value must be multiplied by the probability that this worker was previously employed at a firm of lower productivity, which is \(\frac{N_t(p)}{N_t}\). Finally, under the assumption that only existing workers can be poached, the on-the-job search inflow is multiplied by \((1 - \delta)\).

Defining the distribution of employment across firms as

\[
N_t(p) = \int_p^1 L_t(s) \gamma(s) ds \tag{13}
\]

we can also specify the law of motion for the distribution of employment as:

\[
N_{t+1}(p) = (1 - \delta) \left( 1 - \Phi_t^N \frac{V_t(p)}{V_t} \right) N_t(p) + U_t \Phi_t^U \frac{V_t(p)}{V_t}. \tag{14}
\]

We assume that wages are determined through Nash wage bargaining that maximize the weighted surplus of the worker and firm. The wage rate paid by a type-\(p\) firm, \(W_t(p)\), is the solution to the following optimization problem:

\[
W_t(p) = \arg \max_{W_t(p)} \left( V_t^N(p) - V_t^U \right) \xi \left( V_t^I(p) - V_t^V(p) \right)^{1-\xi}, \tag{15}
\]

where \(\xi\) is the bargaining power of the worker, \(V_t^N(p)\) is the value of being employed at a type-\(p\) firm for a worker, \(V_t^U\) is the value of being unemployed (which does not depend on \(p\)), \(V_t^I(p)\) is the value of employing a worker for a type-\(p\) firm, and finally \(V_t^V(p)\) is the value of posting a vacancy for a type-\(p\) firm. The solution to this Nash wage bargaining problem gives the following implicit equation for wages:

\[
\xi \left( V_t^I(p) - V_t^V(p) \right) = (1 - \xi) \left( V_t^N(p) - V_t^U \right). \tag{16}
\]
We now turn to the discussion of the value functions. The value for a worker of being employed at a type-\(p\) firm can be written as follows:

\[
V_t^N(p) = W_t(p) + \beta E_t \left( (1 - \delta) \left( \Phi_{t,p}^N \int \frac{V_t(s)}{V_t(p)} V_{t+1}(s) \gamma(s) ds + (1 - \Phi_{t,p}^N) V_{t+1}^N(p) \right) + \delta V_{t+1}^U \right)
\]  

(17)

where \(\Phi_{t,p}^N = \Phi_t^N \frac{V_t(p)}{V_t}\) is the probability that a worker employed at a type-\(p\) firm finds a job in a firm of higher productivity. The explanation for the value function is similar to the discussion of the equation describing the dynamics of average firm size. A worker employed at a type-\(p\) firm receives a wage \(W_t(p)\) in period \(t\). In period \(t+1\) with probability \(\delta\) they become unemployed and receive continuation value equal to the value of being unemployed, \(V_{t+1}^U\), or with probability \(1 - \delta\) they will remain employed. If they remain employed at the same firm (which occurs with probability \(1 - \Phi_{t,p}^N\)), they will receive continuation value \(V_{t+1}^N(p)\). If they find a better job, through on-the-job search, the expected value of their new employment is calculated as the vacancy posting intensity weighted value of employment of firms of productivity greater than \(p\).

The value of being unemployed is given by:

\[
V_t^U = b + \beta E_t \left( \Phi_t^U \int \frac{V_t(s)}{V_t} V_{t+1}(s) \gamma(s) ds + (1 - \Phi_t^U) V_{t+1}^U \right)
\]  

(18)

where \(b\) is the exogenous utility flow from being unemployed. The value of being unemployed consists of current unemployment benefits and the discounted expected value of either being employed in the next period, \(V_{t+1}^N(p)\), or the future value of continued unemployment. The latter occurs with the probability \(1 - \Phi_t^U\).

The value for a type-\(p\) firm of employing a worker is equal to:

\[
V_t^J(p) = (A_t p - W_t(p)) + \beta E_t \left( (1 - \delta) \left( 1 - \Phi_t^N \frac{V_t(p)}{V_t} \right) V_{t+1}^J(p) + \delta V_{t+1}^Y \right)
\]  

(19)

The value of employing a worker is equal to the current revenue generated by the worker, \(A_t p\), minus the wage paid, \(W_t(p)\), and the continuation value of employment multiplied by the probability of the job relationship not being severed. The exogenous probability of the job relationship being severed is given by \(\delta\), whereas the probability of the worker switching jobs to a better firm is given by \(\Phi_t^N\), time the probability that the worker is still employed.

Finally, the value of posting a vacancy is given by:

\[
V_t^V(p) = -\Xi(V_t(p)) + \beta E_t \left( \Psi_t^U + \Psi_t^N \frac{V_t(p)}{V_t} \right) V_{t+1}^J(p)
\]  

(20)
that is by the discounted expected future value of employing a worker multiplied by the probability of employing a worker net of the marginal cost of posting the vacancy, $\Xi(V_t(p))$. The probability of employing a previously unemployed job seeker is given by $\Psi^U_t$, whereas the probability of employing an employed job seeker is given by $\Psi^N_t$, corrected for the probability that the job seeker was employed by a worse firm (which is again given by the distribution of employment across firms). Finally, the number of vacancies posted by any firm is given by the free entry condition, namely $V_t^V(p) = 0$.

3 Solution method

To solve the model we introduce the following two-step numerical procedure. In the first step we approximate all functions of firm productivity using Chebyshev polynomial approximation. The functions that need to be approximated are in particular the firm and worker value functions $V_t^X(p)$, wage function $W_t(p)$, vacancy intensity posting $V_t(p)$, average firm size $L_t(p)$ and distribution of employment $N_t(p)$. Following Boyd (2001), the basic $N$-point approximation $P_N^f$ for a function $f(p)$ can be expressed as

$$P_N^f(p) \approx \sum_{n=0}^{N} b_n^f \times T_n(h(p)), \quad (21)$$

where $b_n^f$ are Chebyshev weights, $T_n$ is the $n$-th Chebyshev polynomial and $h(p)$ is a linear transformation between the interval $[\underline{p}, \overline{p}]$ to the canonical interval $[-1, 1]$. The approximation weights are set by the collocation method with Gauss-Lobatto nodes. The nodes $y_k$ are set as follows:

$$x_k = -\cos\left(\frac{k\pi}{N}\right) \quad y_k = h^{-1}(x_k), \quad (22)$$

whereas the weights are defined as follows;

$$b_n^f = \sum_{k=0}^{N} f(y_k)T_{k,n} \quad T_{k,n} = \frac{2\delta_k \delta_n T_n(x_k)}{N}, \quad (23)$$

where $\delta_k = \frac{1}{2}$ for $k = 0, N$ or $\delta_k = 1$ otherwise. Therefore, in order to calculate the value of function $f(p)$ at any point it is enough to know the values at the nodes. This property of Chebyshev polynomial approximation is central to our solution method - every function of productivity can be represented by its value at $N$ nodes.

Solving the model also requires calculating a number of integrals of these functions, such as those that appear in equations (1), (2) or (17). The limits of this integration can be

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3See Antosiewicz (2016) for more detailed description of this procedure.
either the endpoints of the interval \([p, q]\) or any point inside the interval. However, a closer inspection of the equations that contain an integral taken together with the fact that we are only seeking to calculate the values of functions at the nodes \(y_k\) reveals that in the solution method the limits of the integration will only be the nodes \(y_k\). Therefore, in order to calculate an integral of the function \(f(p)\), we use the following scheme:\(^4\)

\[
\int_{-1}^{a} f_t(x) dx \approx \int_{-1}^{a} P_N^t(x) dx = \sum_{n=0}^{N} b_n^t \int_{-1}^{a} T_n(x) dx = \sum_{n=0}^{N} \frac{2\delta_n}{N} \sum_{k=0}^{N} f(x_k) T_n(x_k) \int_{-1}^{a} T_n(x) dx
\]

(24)

(25)

Using a matrix multiplication view, the integral can be written as follows:

\[
\int_{-1}^{a} f_t(x) dx \approx F_t^f \times W \times T = F_t^f \times P
\]

(26)

where

- \(F_t^f\) is a row vector of length \(N + 1\) which contains the values of the function \(f\),
- \(W\) is an \((N + 1) \times (N + 1)\) matrix which contains the values of consequent Chebyshev polynomials in consequent nodes,
- \(T\) is a column vector of length \(N + 1\) which contains the integrals over the interval \([-1, a]\) of consequent Chebyshev polynomials,
- \(P\) is a column vector of length \(N + 1\) which is the result of multiplication of \(W\) and \(T\).

Notice that the matrices \(W\) and \(T\) do not depend either on time or on the function \(f\), and hence they can be calculated analytically outside of the model. The key result that we obtain from using Chebyshev approximations is that all the integrals that appear in the model equations can be replaced by a scalar product of the values of the integrand function in the nodes and exogenously calculated parameters. Finally, we transform the equations of the model into a system of equations in which integrals are replaced by their Chebyshev approximations.

In the second step we take this system of equations and solve (i) for the deterministic steady state and (ii) for the dynamics using a first order log-linearisation around the steady state. The results shown in the next section are based on a model solved using a 20-point approximation (\(N = 19\)), which requires the external calculation of approx

\(^4\)In order to make the exposition clear, we assume the canonical interval without the need to use linear transformation of intervals.
\((N + 1)^2\) parameters for the integrals embedded in the equations of the model. The key advantage of using this solution method is that it allows for easy calculation of model dynamics and facilitates the implementation of various elements (frictions, capital etc.) which are of interest for the researcher.

4 Parametrisation and results

In this section we discuss the baseline parametrisation and then discuss the properties of the model. We proceed as follows: we first document model fit to empirical moments and the behaviour of wages. Next we analyse the distribution of employment and the dynamics of labour turnover across firms. We conclude with a sensitivity analysis with respect to key model parameters.

4.1 Baseline parametrisation

In the baseline parametrisation we follow closely the approach presented in Moscarini and Postel-Vinay (2016). While some changes to the parametrisation are required due to the different assumption of wage setting, the Nash wage bargaining model exhibits similar challenges as the wage-posting counterpart and requires several unconventional parameter values.

Similarly to Mortenen (2003) and Moscarini and Postel-Vinay (2016) we assume that firms are distributed according to a truncated Pareto distribution with exponent equal to 2.

\[
\Gamma(p) = \frac{\alpha}{p^{1+1}} \Gamma^*,
\]

where \(\Gamma^*\) is the normalization coefficient due to the truncation of the distribution to a finite interval.\(^5\) For the vacancy posting cost we assume the following:

\[
c(V) = \nu_1 V^{\nu_2},
\]

which results in the following form for \(\Xi(V_i(p))\):

\[
\Xi(V_i(p)) = c'(V_i(p)) = \nu_1 \nu_2 V_i(p)^{\nu_2-1}.
\]

The parameters of the model are summarized in Table 1. The discount factor is set to 0.99, which implies that the time period in the model is a quarter. The search intensity of unemployed members of the household is normalized to unity, whereas that of employed

\(^5\)Mortensen (2003) uses \(\alpha = 2\).
Table 1: Model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Baseline value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>Pareto exponent</td>
<td>2.5</td>
</tr>
<tr>
<td>$\beta$</td>
<td>discount factor</td>
<td>0.99</td>
</tr>
<tr>
<td>$\delta$</td>
<td>ss job destruction rate</td>
<td>0.049</td>
</tr>
<tr>
<td>$b$</td>
<td>utility of unemployed</td>
<td>0</td>
</tr>
<tr>
<td>$v$</td>
<td>matching function efficiency</td>
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</tr>
<tr>
<td>$\mu$</td>
<td>match elasticity wrt vacancies</td>
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<tr>
<td>$\xi$</td>
<td>bargaining power</td>
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<td>$\nu_1$</td>
<td>vacancy cost scale</td>
<td>43</td>
</tr>
<tr>
<td>$\nu_2$</td>
<td>vacancy cost exponent</td>
<td>50</td>
</tr>
<tr>
<td>$\lambda^e$</td>
<td>search intensity of employed</td>
<td>0.3</td>
</tr>
<tr>
<td>$\lambda^u$</td>
<td>search intensity of unemployed</td>
<td>1</td>
</tr>
</tbody>
</table>

is set to 0.3. The value of this parameter is used to set an average employer to employer quarterly flow of 3.6%, which corresponds to the monthly value of 1.2% reported in the MPV model. Consistently with literature, the steady state job destruction rate is set to 0.049 but following MPV we assume that it fluctuates along with aggregate technology $A_t$ according to:

$$\delta(A_t) = 0.0204 + 3.285 \times (0.15 - \log(A_t))^{2.5}. \quad (30)$$

The aggregate productivity shock $A_t$ follows an AR(1) process with autocorrelation coefficient equal to 0.95 and a standard deviation of 0.01. The scale of the matching function $v$ is normalized to unity, whereas match elasticity with respect to vacancies is set to 1, implying a linear function of vacancies. The non-conventional choice of elasticity with respect to vacancies is justified by the fact that total matches are a function of total job search effort, which also takes into account employed job seekers, whereas empirical estimates of matching functions usually take into account only the unemployed. The choice for the remaining parameter values, which are those governing the vacancy posting cost, wage bargaining power of workers and utility of unemployed, accounts for the fact that with standard values there is a tendency for firms of low productivity to go out of business during downturns and sometimes even in the steady state. In the on-the-job search setup of the model, firms of higher productivity poach workers from firms of lower productivity, and under some parameter values the value of a job for those firms falls below zero. For
Table 2: Cyclical behaviour of the model: standard deviations and cross-correlations of aggregate variables.

<table>
<thead>
<tr>
<th></th>
<th>US data</th>
<th>model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unemployment rate</td>
<td>0.216</td>
<td>0.088</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U \rightarrow E$</td>
<td>-0.974</td>
<td>0.121</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E \rightarrow U$</td>
<td>0.889</td>
<td>-0.887</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tightness</td>
<td>-0.978</td>
<td>0.972</td>
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<tr>
<td></td>
<td>-1.00</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average labour</td>
<td>0.108</td>
<td>-0.017</td>
</tr>
<tr>
<td>Productivity (ALP)</td>
<td>-0.939</td>
<td>0.937</td>
</tr>
</tbody>
</table>

Note: Standard deviations (diagonal) and cross-correlations.

example, for the case of linear vacancy posting costs, high productivity firms post so many vacancies that workers do not stay long enough in low productivity firms to offset the cost of posting the vacancy. A similar effect can be observed for parameter $b$ and $\xi$. Increasing both parameters results in an increase in the wage$^6$, which also has a tendency to push the lowest type-$p$ firms out of the market, since they are the most vulnerable to wage increases. This is all the more evident when one examines the difference between productivity of firms and the wage for different productivity levels. For a firm with productivity $p = 1$ the wage is 0.88, while for a firm with $p = 10$ it is 3.8.

4.2 Basic model properties

In this subsection we document the basic business cycle properties of the model. We start with an analysis of the fit of the model to empirical moments for main labour market variables. Table 2 shows the standard deviations (which are on the diagonal) and correlations of the unemployment rate, worker flows, tightness and average labour productivity (ALP). The statistics are taken from MPV (2016) and show the HP-filtered quarterly statistics for the USA economy. The model is successful in generating a significantly larger volatility of the unemployment rate and labour market tightness comparing to ALP, but these values still fall short of those reported in the data and the stochastic wage posting model. The

$^6$Parameter $b$ increases the surplus of a worker, and hence the negotiated wage, whereas $\xi$ increases the wage directly.
Table 3: Correlation with ALP for other uncorrelated shocks and shock amplification.

<table>
<thead>
<tr>
<th></th>
<th>U rate</th>
<th>UE rate</th>
<th>tightness</th>
<th>amplification</th>
</tr>
</thead>
<tbody>
<tr>
<td>data</td>
<td>0.108</td>
<td>-0.017</td>
<td>-0.011</td>
<td>16.6</td>
</tr>
<tr>
<td>matching technology</td>
<td>-0.444</td>
<td>0.070</td>
<td>0.445</td>
<td>14.8</td>
</tr>
<tr>
<td>job destruction rate</td>
<td>-0.456</td>
<td>0.461</td>
<td>0.457</td>
<td>14.5</td>
</tr>
<tr>
<td>discount factor</td>
<td>0.805</td>
<td>-0.492</td>
<td>-0.625</td>
<td>2.5</td>
</tr>
</tbody>
</table>

lower degree of amplification is due to the replacement of wage setting mechanism with Nash wage bargaining. Under this assumption, an increase in productivity automatically translates to an increase of wages, which depresses the intensity of vacancy posting by firms. The correlations between labour market variables are replicated by the model quite closely, with the exception of ALP. In their article, MPV (2016) suggest including additional uncorrelated shocks to enhance the properties of the model in this dimension, such as a shock to the matching function, job destruction rate or discount factor. The results for this exercise are presented in Table 3. It presents the correlations with labour market variables (except for employment to unemployment flows, which in case of these shocks are constant) and the amplification of the shock, which is measured as the relative standard deviation of the unemployment rate and ALP. It appears that uncorrelated shocks would help, however with most correlations at the level of approximately 0.5 it would only be to a limited extent. The discount rate shock would go the furthest in improving model statistics, since the correlations are of the opposite sign, whereas the matching technology would reduce the correlation with UE flows to some degree. However, there is another dimension along which the matching and destruction rate shocks would improve model properties. Since these are typical ‘labour market’ shocks, they have a relatively strong impact on the behaviour of unemployment, and little effect on average labour productivity. There are few other uncorrelated shocks one could consider, for example shocks to the search intensity of both employed and unemployed. The addition of these would help in terms of model properties, however, as Leyva (2016) and Mukoyama et al. (2014) show, aggregate search effort changes very little over the cycle.

4.3 Behaviour of wages

Following MPV (2016), in order to analyse the cyclical behaviour of wages we run a regression of log wages on the unemployment rate. The computed model semi-elasticity is equal to $-3.59$, which is much more than the value of $-0.6$ that MPV report for the
USA economy. While this is a value that is half that of the wage posting model and significantly closer to the data, it is important to keep in mind that these statistics are for a model which underdelivers in terms of the volatility of unemployment.

A surprising feature of the Nash wage bargaining model can be observed in Figure
1, which shows the standard deviation of the log surplus of the worker-firm match and of wages. For most productivity levels, the standard deviation of the surplus is higher than the standard deviation of wages. This is the opposite result than in the case of the wage posting model. Wages are still more cyclical than in the data, however this feature shows that wage bargaining has more potential in terms of matching their volatility. It is also interesting to note that, in line with the data, the standard deviation of log wages is higher, although only by a factor of 1.5 for the highest wage than for the lowest.

Figure 2 shows the distribution of wages. Majority of workers earn small and medium wages with relatively few workers earning high wages. The ratio between the largest and smallest wage is equal to 4.3. While this value is smaller than what is observed in real life, this result is unsurprising given a linear production function of labour and firm productivity varying between 1 and 10. On the other hand, regarding cross-sectional wage dispersion, the Nash wage bargaining model does a much better job. Contrary to the wage posting model, Nash wage bargaining overstates the residual wage dispersion reported by MPV, which is 0.18. The model generates a cross-section wage variance of approximately 0.21, and furthermore the wage bargaining power parameter can be used to calibrate the model’s variance of wages exactly to the data. This flexibility of the model is due to the fact that wages have a relatively weak impact on the behaviour of firms and on labour turnover. In the wage posting mechanism, workers reallocate to firms of higher wages and firms take wages and the resulting poaching of workers into account. Finally, the Gini coefficient for the distribution of wages is approximately 0.174, which is much lower than for the USA, however this value is only slightly lower than for the most equal OECD countries.

4.4 Distribution and fluctuations of employment across firms

Figure 3 shows the steady state distribution (shown as density function) of employment across firms which is plotted against the truncated Pareto distribution. The two plots are aligned, however firms of the lowest productivity employ relatively fewer workers then firms of higher productivity. The density for the lowest $p$-firms is only slightly decreasing, which results from the fact that the decreasing Pareto density and increasing firm size counterbalance themselves. However, for larger values of productivity, the Pareto density of firms overweighs the increasing average firm size which results in a very small density of total employment for the right tail.

As shown in Figure 4 the main properties of the Burdett and Mortensen (1998) model hold—the wage and average firm size increase along with productivity. The average firm size is a concave function of productivity, which is a direct result of the high convex cost of vacancy posting which prevents large firms from accumulating a large number of workers.
Figure 3: Steady state distribution of workers across firms (solid line) and density of firm productivity (dashed line)

Figure 4: Firm size in terms of (a) employment or (b) profits.

This also results in a relatively small ratio between the largest and smallest firm measured in employment, which is approximately 15, whereas in the data this value is several orders of magnitude larger. In the MPV (2016) baseline calibration this value is much lower at 4.24. However, the concavity of the firm size-productivity relationship helps the model replicate the data along a different dimension. Figure 4 plots the density function of employment across different firm sizes rather than productivity. It features relatively fat right tail which is in line with data regarding the distribution of employment from the Business Dynamics Statistics database for the United States. As can be seen from Figure 4b, firms differ much more with respect to profits, with the difference between most and least profitable firm being approximately 2500. This is a result of the fact that
the surpluses of the firm and worker are very small for low-productivity firms, driving wages close to their productivity level.

The wage bargaining model does a very good job in replicating the differences in cyclical behaviour between gross and net job creation. Figure ?? shows the dynamic responses of the distribution of employment in response to an aggregate productivity shock \(A_t\) 4 and 12 quarters after the shock hits. Overall, the model predicts a shift in the distribution of workers to the right, that is in good times of low unemployment large firms fare relatively better. In absolute terms, the 1% increase in productivity is a stronger incentive for high-\(p\) firms to post additional vacancies for two reasons. First, there is a differences in the size of the wedge between marginal productivity and the wage they pay to workers. This wedge is very small for low productive firms and large for highly productive firms, as shown in Figure ?? . Large firms can easily cope with the increase in wages which result from an increase in productivity and worker values, whereas low productivity firms operates on the edge of profitability. Secondly, an increase in vacancy posting by large firms implies more job losses for smaller firms operating at the lower end of the productivity distribution, which then reduces incentives for them to post vacancies. This feature is consistent with the empirical observation that large firms contribute more to net employment in good times of low unemployment.

On the other hand, as we show in Figure ?? , the total number of gross hires is much more volatile at the bottom of the distribution, which is another key dimension along which the model does well. This fact has been documented among other in Moscarini and Postel-Vinay (2012) and Haltiwanger et al. (2014). In the model, since small firms face increased poaching from more productive high wage firms, they can only rely on the pool of unemployed for new workers and, hence, they increase hiring. However, the number of workers they lose to poaching from firms of higher productivity outweighs these additional hires, resulting in difference in net hires.

This mechanism is evident analysing the steady state employment flows between firms. We first consider the inflow of workers to firms and look at how many workers are poached from other firms relative to hires from the pool of the unemployed. The results for this ratio is shown in Figure ?? . The plot starts at the value zero, as firms of the lowest possible productivity do not have the possibility to poach any workers, and rises to up to almost 3 for the most productive firm. It implies that approximately 75% of new hires are poached from other firms. Figure ?? presents the rate at which firms are losing workers to other firms through on the job search. The smallest firms are losing almost 15% of their workforce quarterly to poaches from other firms. The on-the-job search separation rate quickly reaches zero for medium productivity firms and remains at this level. This result is driven by the highly convex vacancy posting cost.
Figure 5: Change in distribution of employment in response to productivity shock with constant job destruction rate

The panels of Figure ?? show the arrival rate for hires from the pool of the unemployed and hires from other firms. The picture here is slightly more complicated than in the case of on the job search separations, since in order to achieve an equilibrium, these rates have to implicitly take into account the job losses due to poaching. The hiring rate is a monotonically decreasing function of productivity. Since low-

\( p \)

firms rely mainly on hires unemployed and lose a lot of workers due to poaching, their hiring rate has to be high. On the other hand, the on-the-job arrival rate is not monotonic as for higher productivity firms it becomes a decreasing function. This is due to the fact that for higher type-

\( p \)

firms the decreasing probability of having a worker poached has a relatively stronger effect than the need for new hires (in equilibrium).

Finally, we test the extent to which model properties rely on the assumption that the job destruction rate is a function of aggregate productivity, see equation (30). In Figure 5 we plot the response of the distribution of firms to a productivity shock for which we keep the job destruction rate constant. The overall conclusion is that most of the amplification in the aggregate productivity shock comes from the fact that the job destruction rate is correlated with the aggregate productivity shock. Whether this is also the case for MPV (2016) model it is not clear, since authors do not report these findings.

The second shock for which we show the response of the distribution of employment is the job destruction rate shock which is shown on Figure 6. Since low-

\( p \)

firms incur
smaller costs of posting vacancies, an increase in the job destruction rate hurts them less and shifts the distribution of workers to the left. This dynamic model property is also in line with empirical facts regarding job creation, since, as one would expect, the job destruction rate is negatively correlated with labour productivity.

### 4.5 Sensitivity analysis

This subsection shows the sensitivity analysis of the steady state distribution of the distribution of employment under different values of selected parameters. Figures ?? and ?? show the steady state distribution of employment across firms for various values of the Pareto exponent and of the steady state job destruction rate.

The effect of the Pareto exponent parameter can be seen on Figure ???. As can be expected, the overall effect is that a higher density of more productive firms translates into higher overall employment at larger firms. Finally, the sensitivity analysis for the steady state job destruction rate is shown in Figure ???. Higher job destruction rate results in more small, low-$p$ firms due to the fact that they face smaller vacancy posting costs.

### 5 Conclusions

In this paper we consider a stochastic version of on-the-job search model a la Burdett-Mortensen (1998) based on Moscarini and Postel-Vinay (2016). The contribution of this
paper is two-fold. First of all, we proposed a novel and elastic technique to solve and then simulate the model that uses both projection and perturbation methods. Secondly, we construct a job ladder model with on the job search and Nash wage bargaining and compare its properties against the wage posting counterpart. We analyse the model’s ability to replicate empirical moments, the steady state distribution of labour market variables and then show how the aggregate shocks affect the flows of workers across firms. The main success of the model is that it is able to account for the dynamics of net and gross job creation among firms of different size. We also find that using wage bargaining instead of wage posting helps along certain dimensions such as predicted wage dispersion. On the other hand it fails to reproduce sufficient amplification of shocks and volatility of labour market variables, a feature which is known for wage bargaining models. However, as the numerical method is fairly flexible it allows for effectively conducting sensitivity analysis without the need for computationally extensive simulations for each parameter value. The model can also be easily expanded along various dimensions such as the incorporation of capital into the production function. This model extension can help in two dimensions: we expect that it should be possible to generate a more realistic ratio between the largest and smallest firm, and a more realistic ratio between top and bottom earners. Finally, the method can also be used to solve a wide array of dynamic stochastic general equilibrium models which incorporate heterogeneity.
References


